Advanced ODE-Lecture 2
Ascoli-Arzela Lemma and Inequalities
Existence and Uniqueness

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Outline

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- **Ascoli-Arzela Lemma**
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Motivation

$$\text{IVP:} \quad \begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

- Ascoli-Arzela Lemma is a key tool to show Peano theorem for solution existence of IVP, no matter what ways you employ to prove this result.

- Inequality techniques, geometric approaches and Lyapunov methods are some powerful tools in analysis of ODE. Inequality techniques are used very often. Gronwall inequality is one of basics.

- Peano Theorem assures solution existence of IVP.

Ascoli - Arzela Lemma

Ascoli-Arzela Lemma: Let $F = \{ f : [a,b] \to R^n \}$ be **a family of functions** on $[a, b]$ such that

- $F = \{f\}$ is **equicontinuous**;
- $F = \{f\}$ is **uniformly bounded**

Then the family $F = \{f\}$ has a convergent subsequence $\{g_n\} \in F$, which converges uniformly to $g \in C([a, b])$.

Remark 2.1 A bounded closed set in $C([a,b])$ is not necessarily compact! However, a uniformly bounded set of equicontinuous functions in $C([a,b])$ is compact! So Ascoli-Arzela lemma can be regarded as a generalization of Bolzano-Weierstrass theorem in R^n to $C([a,b])$.

Remark 2.2

1) $F=\{f\}$ is **equicontinuous** on $[a,b]$ if $\forall\,\varepsilon>0$, there exists $\delta>0$, s.t. for any $f\in F$ and any $t_1,t_2\in[a,b]$, we have $\| f(t_1)-f(t_2)\|<\varepsilon$ whenever $|t_1-t_2|<\delta$;

2) $F=\{f\}$ is **uniformly bounded** on $[a,b]$ if there exists $M>0$, independent of choice of f in F s.t. for any $f\in F$, implies $\| f(t)\|\le M$, $t\in[a,b]$.

Remark 2.3 $F=\{f\}$ is not necessarily countable.

Gronwall Inequality

Gronwall Inequality Suppose that $u(t), h(t) \in C([a,b])$ with $h(t) \geq 0$ and $C \geq 0$.

If

$$u(t) \leq C + \int_a^t u(s)h(s)ds, \quad t \in [a,b],$$

then, one has

$$u(t) \leq Ce^{\int_a^t h(s)ds}, \quad t \in [a,b].$$

Proof. Let $g(t) = C + \int_a^t u(s)h(s)ds$, then $u(t) \leq g(t)$, $t \in [a,b]$.

$\Rightarrow$ $g(t)$ is smooth and $g'(t) = u(t)h(t) \leq g(t)h(t)$ by $h(t) \geq 0$. Set

$$v(t) := g'(t) - g(t)h(t).$$

Then $v(t) \le 0$. We solve the linear initial problem as follows:

$$g'(t) = g(t)h(t) + v(t), \text{ with } g(a) = C$$

to obtain

$$g(t) = e^{\int_a^t h(s)ds}[C + \int_a^t v(s)e^{-\int_a^s h(\tau)d\tau}ds] \le Ce^{\int_a^t h(s)ds},$$

which yields $u(t) \le g(t) \le Ce^{\int_a^t h(s)ds}$. □

Remark 2.4 Gronwall inequality has several ways to prove. You may do some others by yourself. The above way just takes the advantage of the solution formula of linear differential equations. However, how to apply this inequality effectively is our concern.

Peano Theorem

Consider the IVP given by

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases},$$

where $f: Q \to R^n$ is continuous, $Q = \{(t,x) \in R \times R^n : |t - t_0| \le a, \ \|x - x_0\| \le b\}$.

1) Statement of Peano Theorem

Peano Theorem Suppose that $f: Q \to R^n$ is continuous. Then, the IVP has a solution $x(t)$ on $I = [t_0 - h, t_0 + h]$, where $h = \min\{a, \frac{b}{M}\}$, $M = \max_{(t,x)\in Q} \| f(t,x) \|$.

2) Proof of Peano Theorem (by Schauder Fixed Point Theorem)

Step 1. Define a Banach space $C(I)$, where $I=[t_0-h,t_0+h]$ with a norm given by

$\| x \|_\infty = \max_{t\in I} \| x(t) \|, \quad \forall x \in C(I)$.

Step 2. Define a closed subset of $C(I)$ as follows

$$D=\{x : x\in C(I),\ \| x-x_0 \|_\infty \le b,\ t\in I\} \subset C(I).$$

Show D is convex. For any $x_1, x_2 \in D$, one has

$$\| \lambda(x_1-x_0)+(1-\lambda)(x_2-x_0) \|_\infty \le \lambda \| x_1-x_0 \|_\infty + (1-\lambda)\| (x_2-x_0) \|_\infty \le \lambda b+(1-\lambda)b=b,$$

$$\Rightarrow \quad \lambda(x_1-x_0)+(1-\lambda)(x_2-x_0)\in D. \quad \Rightarrow \quad D \text{ is convex.}$$

Therefore, D is a convex closed subset of $C(I)$.

Step 3. Define a mapping $T : D \to D$ as follows.

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds\,, \quad \forall\, t \in I\,, \quad \forall\, x \in D\,.$$

To show $T(D) \subset D$, since

$$\| (Tx)(t) - x_0 \| \le \int_{t_0}^{t} \| f(s, x(s)) \| ds \le M \,| t - t_0 | \le M\, \overline{h} \le M\, h \le b, \;\; \forall t \in \overline{I}\,, \;\; \forall x \in D\,,$$

and $\overline{I}$ is compact, then $\max_{t \in \overline{I}} \| (Tx)(t) - x_0 \| \le b$. i.e. $\| Tx - x_0 \|_\infty \le b$, which implies $Tx \in D$. Therefore, $T(D) \subseteq D$.

To show that T **is completely continuous on** D, following the definition, we show that T is continuous on any bounded set $D_1 \subset D$ and $T(D_1)$ is relatively compact in D. Since D is a bounded set, so we take $D_1 = D$ without loss of generality.

Step 4. Show that $T(D)$ is relatively compact in D.

Being a family of functions defined on I, if T is uniformly bounded and equicontinuous, then there exists a convergent subsequence in D by Ascoli-Arzela lemma. So $T(D)$ will be relatively compact in D. The remaining is to show T is uniformly bounded and equicontinuous.

Since $T(D) \subset D$, i.e. $\| Tx - x_0 \|_\infty \le b \Rightarrow \| Tx \|_\infty \le \| x_0 \|_\infty + b$, so T is uniformly bounded.

For $\forall\ \varepsilon > 0$, taking $\delta = \frac{\varepsilon}{M} > 0$, for any $x \in D$ and any $t_1, t_2 \in I$, if $| t_1 - t_2 | < \delta$, we have

$$\| Tx(t_2) - Tx(t_1) \| \le \| \int_{t_1}^{t_2} f(s, x(s)) ds \| \le M \, | t_2 - t_1 | < \varepsilon ,$$

T is equicontinuous, which is obviously continuous.

Applying Ascoli-Arzela lemma yields that $T(D)$ has a convergent subsequences in D. This shows that $T(D)$ is relatively compact in D by definition. Therefore, T is completely continuous on D.

Sept 5. Conclusion of solution existence.

Since all the conditions are satisfied, it is concluded that there exists (not necessarily unique) a fixed point $x^* \in D$ s.t. $Tx^* = x^*$ by Schauder fixed point theorem. That is,

$$x^*(t) = x_0 + \int_{t_0}^{t} f(s, x^*(s))ds, \ t \in I .$$

This shows that the IVP has a solution $x^*(t)$ on I. □

3) Application of Peano Theorem

Remark 2.5 Based on Peano theorem, we can show Picard theorem in a simple and different way.

Statement of Picard Theorem Suppose that $f : Q \to R^n$

- is continuous;
- satisfies a Lipschitz condition as follows:

$$\| f(t, x) - f(t, y) \| \le L \| x - y \|, \text{ for any } (t, x), (t, y) \in Q.$$

Then the IVP has a unique solution $x(t)$ at least defined on $I = [t_0 - h, t_0 + h]$, where

$$h = \min\{a, \frac{b}{M}\}, \quad M = \max_{(t,x)\in Q} \| f(t, x) \|.$$

Proof. The existence is obtained by Peano Theorem.

For uniqueness, if there exist two solutions $x_1(t)$ and $x_2(t)$ s.t.

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))ds, \ \ t \in I;$$

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_2(s))ds, \ \ t \in I.$$

Subtracting them yields

$$\| x_1(t) - x_2(t) \| = \left| \int_{t_0}^{t} \| f(s, x_1(s)) - f(s, x_2(s)) \| ds \right|$$

$$\le L \left| \int_{t_0}^{t} \| x_1(s) - x_2(s) \| ds \right|.$$

Without loss of generality, we assume that $t \in I^+ = [t_0, t_0 + h]$. Then,

$$\| x_1(t) - x_2(t) \| \le L \int_{t_0}^{t} \| x_1(s) - x_2(s) \| ds.$$

Without loss of generality, we assume that $t \in I^+ = [t_0, t_0 + h]$. Then,

$$\| x_1(t) - x_2(t) \| \le L \int_{t_0}^{t} \| x_1(s) - x_2(s) \| ds .$$

Application of Gronwall inequality results in $\| x_1(t) - x_2(t) \| \le 0$, which is not possible unless $x_1(t) \equiv x_2(t)$, $t \in I^+$. It is similar for $t \in I^- = [t_0 - h, t_0]$. The uniqueness is proved. □

Remark 2.6 There are several ways to show the uniqueness. You are encouraged to find more by yourselves

Remark 2.7 Either Peano Theorem or Picard Theorem is local result since $Q \in R^{n+1}$ is compact (bounded and closed).

Summary

1) Continuity $\Rightarrow$ Existence (Peano Theorem).

2) Lipschitz Condition $\Rightarrow$ Uniqueness (Gronwall Inequality).

3) Continuity + Lipschitz Condition $\Rightarrow$ Existence and Uniqueness (Picard Theorem).

4) Peano Theorem and Picard Theorem are local results.

Homework

1) Suppose that $g(t), C(t), \ h(t) \in C([a,b])$ with $C(t) \geq 0$ and $h(t) \geq 0$. If

$$g(t) \leq C(t) + \int_a^t g(s)h(s)ds, \quad t \in [a,b],$$

then

$$g(t) \leq C(t) + \int_a^t h(s)C(s)e^{\int_s^t h(\tau)d\tau} ds, \quad t \in [a,b].$$

2) Show Picard Theorem by Banach fixed point theorem.

3) Show Ascoli-Arzela Lemma.

4) Show Peano Theorem by a traditional way.


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